

**\*\*FULL TITLE\*\***

*ASP Conference Series, Vol. \*\*VOLUME\*\*, \*\*YEAR OF PUBLICATION\*\**

**\*\*NAMES OF EDITORS\*\***

## Multigrid Methods for Polarized Radiative Transfer

J. Štěpán

*Astronomical Institute, Academy of Sciences of the Czech Republic,  
CS-25165 Ondřejov, Czech Republic*

*LERMA, Observatoire de Paris-Meudon, 5 Place Jules Janssen,  
F-92195 Meudon, France*

**Abstract.** A new iterative method for non-LTE multilevel polarized radiative transfer in hydrogen lines is presented. Iterative methods (such as the Jacobi method) tend to damp out high-frequency components of the error fast, but converges poorly due to slow reduction of low-frequency components. The idea is to use a set of differently coarsened grids to reduce both the short- and long-period errors. This leads to the so-called multigrid (MG) methods. For the grid of  $N$  spatial points, the number of iterations required to solve a non-LTE transfer problem is of the order of  $O(N)$ . This fact could be of great importance for problems with fine structure and for multi-dimensional models. The efficiency of the so-called standard MG iteration in comparison to Jacobi iteration is shown. The formalism of density matrix is applied to the demonstrative example of 1D, semi-infinite, non-magnetic, 3-principal level hydrogen atmospheric model. The effect of depolarizing collisions with thermal electrons is taken into account as well as general treatment of overlapping profiles.

### 1. Formulation of the Problem

In this paper we briefly discuss the usage of multigrid (MG) iteration schemes to solve the non-LTE problem of the 2nd kind as defined by Landi Degl’Innocenti (1987). The era of extensive development of MG methods started in 1970’s by the work of Brandt (1977). Several steps in using MG methods applied to radiative transfer were made by Steiner (1991), Våth (1994), and Fabiani Bendicho, Trujillo Bueno, & Auer (1997). These authors showed that this technique leads to a great improvement of the convergence rate. This paper demonstrates how to apply these methods to a more general solution of polarized radiative transfer with realistic multi-level atomic models and complicated structure of overlapping lines. The effects of depolarizing collisions is taken into account.

For the description of the atomic state, we adopt the density-matrix formalism and the representation in the basis of irreducible tensorial operators (e.g., Fano 1957). The elements of atomic density matrix have the usual form<sup>1</sup>  $\rho_q^k(\alpha j)$ , where  $\alpha j$  is the energy level of total angular momentum  $j$ , and  $(k, q)$  are the multipolar components of the level ( $k = 0, \dots, 2j$ ,  $q = -k, \dots, k$ ). In sta-

---

<sup>1</sup>We suppose that all coherences between different energy levels vanish due to further assumptions.

tionary regime, the density-matrix elements are solutions of the local statistical equilibrium equations,

$$\sum_{\alpha j k q} \Pi_{\alpha' j' k' q', \alpha j k q} \rho_q^k(\alpha j) = 0. \quad (1)$$

The structure of the  $\Pi$ -matrix has been extensively discussed by Sahal-Br  chot (1977), Bommier (1978), and Bommier (1980). We assume that this matrix has the form  $\mathbf{\Pi} = \mathbf{R} + \mathbf{C}$ , where  $\mathbf{R}$  is the matrix of radiative rates, and  $\mathbf{C}$  is the matrix of collisional rates (impact approximation).

The radiative transfer equation for the set of four Stokes parameters  $\mathbf{S} \equiv (I, Q, U, V)^T$  has the usual form

$$\frac{d\mathbf{S}}{ds} = \mathbf{J} - (\mathbf{K} - \mathbf{K}^s) \mathbf{S},$$

where  $\mathbf{J}$  is the emission vector of the local sources,  $\mathbf{K}$  is the absorption matrix, and  $\mathbf{K}^s$  is the matrix of stimulated emission. All these quantities are dependent on radiation frequency,  $\nu$ , position vector,  $\mathbf{x}$ , and direction of propagation determined by the unit vector  $\mathbf{\Omega}$ . Finally,  $s$  is the parametrization of the radiation path along the  $\mathbf{\Omega}$  direction.

## 2. Standard Multigrid Method

Most of the existing non-LTE solvers use the methods based on  $\Lambda$ -operator splitting similar to the one of Rybicki & Hummer (1991, 1992). Depending on organization, these schemes are numerically equivalent to the Jacobi or Gauss-Seidel smoothing procedures (for details, see Paletou & L  ger 2005). These smoothing procedures do reduce high frequencies of the solution fast, but poor convergence is achieved for low frequencies. (With ‘‘high frequencies’’ we mean those which are comparable to the spatial frequency of grid points approximating the continuous scale.) The principles of MG schemes are based on the idea of using coarse grids to reduce the low frequencies, and fine grids to smooth their high-frequency components. It can be showed that such a process may lead to the optimal CPU time demands of  $O(N)$ ,  $N$  being the number of points per decade of optical scale. For comparison, the Jacobi and Gauss-Seidel methods scales approximately as  $O(N^2)$ . We have applied the non-linear version of the *standard multigrid* scheme based on the coarse grid correction (CGC) technique (for details, see Hackbush 1985).

CGC is the process of correction of the fine-grid approximation of the solution using the solutions on the coarse grids. Schematically, it can be described in the following way: the defect (or residuum) of the fine-grid approximation is computed by several calls of the sweeping procedure (Jacobi, etc.); then both defect and the initial guess of the solution are restricted to the coarse grid, and a new solution on the coarse grid is obtained using these data. This coarse-grid solution is interpolated to the fine grid, and the density-matrix components are corrected. This process can be repeated recursively for every grid in order of increasing grid steps. This recursive process leads to the so-called V, W, or more complicated diagrams, depending on the way in which the recursion is implemented (Hackbush 1985).

### 3. Detailed Model Parameters

We adopted a hydrogen atomic model with 9 fine-structure energy levels ( $1s^{1/2}$  to  $3d^{5/2}$ ) and all the non-vanishing multipole components  $\rho_0^k(nlj)$  of density matrix. All the coherence elements ( $q \neq 0$ ) are identically zero due to the model's symmetries. The coherences between different energy levels have been neglected due to the small natural widths of the levels in comparison to their separations, and due to the selection rules for dipole radiative transitions.

The number of logarithmically spaced nodes in the fine grid is  $N_5 = 257$  and 5 grids were used in total. The number of nodes in the “ $G - 1$ ” grid is equal to  $N_{G-1} = (N_G - 1)/2 + 1$ . The initial guess for the atomic density-matrix elements is given by the LTE populations determined in the unpolarized case.

Figure 1 shows a comparison between the convergence rates of the Jacobi and the MG methods. The effect of coarse-grid solutions is reflected by the dramatic evolution of the maximum relative change of the populations. The coarser the grid the shorter the evaluation time, and the higher the rate of approaching to the truncation error of the grid. The maximum relative error dominated by the long-period components is strongly reduced by the recursive CGC during the V-cycles.

The time saving of the MG method in this particular model is about a factor 4 compared to the Jacobi method. It must be noticed that the efficiency of MG increases as fast as the efficiency of the smoothing procedure. The most important benefit from MG's is the asymptotical  $O(N)$  behavior, which designates the method for use in solutions of problems that necessitate strong refinements. Moreover the presented 1D geometry is the slowest case as pointed out by Steiner (1991) and Fabiani Bendicho et al. (1997).

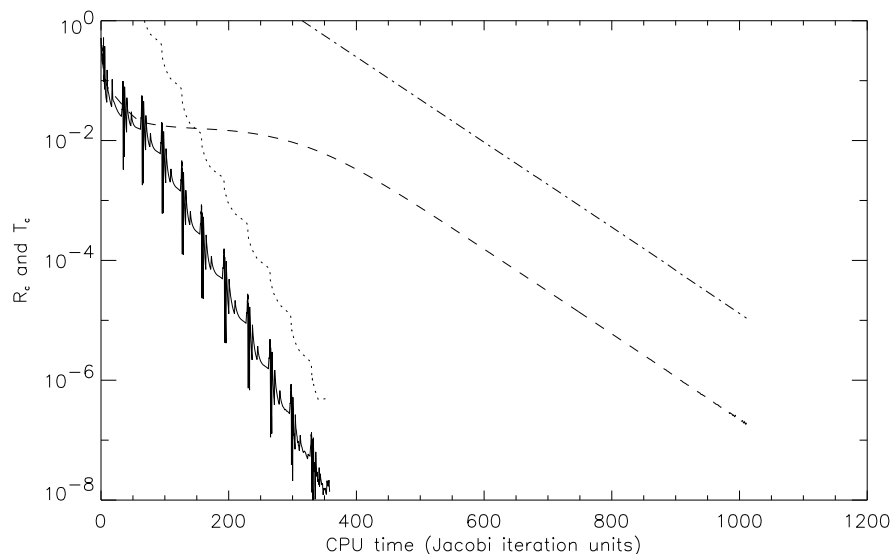


Figure 1. Convergence of the MG method with Jacobi smoothing procedure compared to the Jacobi method. The maximum relative change  $R_e$  of atomic populations,  $\rho_0^0$ , in the MG case (solid line) is compared to the Jacobi solution (dashed line). The maximum relative error  $T_e$  (with respect to the fully converged solution) for MG case (dotted line) and Jacobi case (dot-dashed line) is showed as well. The norm used is  $\|\cdot\|_\infty$  (see Fabiani Bendicho et al. 1997). The graph shows the effect of 11 V-cycles with 2 pre- and 15 post-smoothing Jacobi iterations.

**Acknowledgments.** I'm grateful to my Ph.D. supervisors, Petr Heinzel and Sylvie Sahal-Br  chot, for their useful comments.

## References

- Bommier, V., & Sahal-Br  chot, S. 1978, *A&A*, 69, 57  
 Bommier, V. 1980, *A&A*, 87, 109  
 Brandt, A. 1977, *Math.Comp.*, 31, 333  
 Fabiani Bendicho, P., Trujillo Bueno, J., & Auer, L. 1997, *A&A*, 324, 161  
 Fano, U. 1957, *Rev.Mod.Phys.*, 29, 74  
 Hackbush, W. 1985, *Multi-Grid Methods and Applications* (Berlin: Springer)  
 Kunasz, P., & Auer, L.H. 1988, *JQSRT*, 39, 67.  
 Landi Degl'Innocenti, E. 1987, in *Numerical Radiative Transfer*, ed. W. Kalkofen (Cambridge: Cambridge Univ.), 265  
 Manso Sainz, R., & Trujillo Bueno, J. 2003, in *ASP Conf. Ser. Vol. 307, Solar Polarization 3*, ed. J. Trujillo Bueno & S  nchez Almeida (San Francisco: ASP), 251  
 Paletou, F., & L  ger, L. 2005 (*astro-ph/0507021*)  
 Rybicki, G. B., & Hummer, D. G. 1991, *A&A*, 245, 171  
 Rybicki, G. B., & Hummer, D. G. 1992, *A&A*, 262, 209  
 Sahal-Br  chot, S. 1977, *ApJ*213, 887-899  
 Sahal-Br  chot, S., Vogt, E., Thoraval, S., & Diedhiou, I. 1996, *A&A*, 309, 317  
 Steiner, O. 1991, *A&A*, 242, 290  
 V  th, H. M. 1994, *A&A*, 284, 319